(Model Setup): Global Flights to Safety: An Analysis of the Dollar Liquidity Premium and Cross-border Capital Flows

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1 Model Setup

1.1 Representative Households

1.1.1 Preferences

Households maximize expected lifetime utility: The household maximizes

$$\max_{\{C_t, H_t, L_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t \; - \; \xi \ln(1 - H_t) \; + \; \phi \ln L_t \right],$$

where $0 < \beta < 1$ is the subjective discount factor and $\xi > 0$ governs the Frisch elasticity $\eta = \frac{1}{1+\xi}$ after log-linearization.

- C_t is consumption,
- H_t is hours.

Consumption is a composite of the domestic final good and the imported good.

$$C_t = \left[(1-\gamma)^{\frac{1}{\epsilon}} (HM_t)^{\frac{\epsilon-1}{\epsilon}} + \gamma^{\frac{1}{\epsilon}} (IM_t)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} \tag{1}$$

From expenditure duality, the ideal price index for the aggregator is:

$$P_t = \left[(1-\lambda)(P_t^H)^{1-\varepsilon} + \lambda (P_t^F)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} \tag{2}$$

For liquidity services L_t we have:

$$L_t = \left[(1-\theta)\, M_t^{\frac{\zeta-1}{\zeta}} + \theta \, (B_t^{US}/P_t)^{\frac{\zeta-1}{\zeta}} \right]^{\frac{\zeta}{\zeta-1}}. \tag{3}$$

1.1.2 Budget Constraint (in \$)

Setting the domestic price level to one and denoting the gross real returns as

$$R_t^{US} = 1 + r_t^{US} + \psi_t, \qquad R_t^F = (1 + r_t^F)\,\frac{S_{t+1}}{S_t},$$

we need to change the $-\psi_t$ to a $+\psi_t$, because otherwise, if $R_t^{US} = R_t^F$ in equilibrium, then $r_t^{US} > r_t^F$, which is the opposite of reality. US treasuries pay among the least interest because of the convenience yield on the dollar. Hence the convenience yield needs to be positive. the period-t constraint becomes

$$P_tC_t + P_tI_t + M_t - M_{t-1} + \frac{B_{t+1}^{US}}{R_t^{US}} + \frac{S_t\,P_t^F b_{t+1}^F}{R_t^F} \;=\; W_t\,H_t + R_{k,t}\,K_{t-1} + B_t^{US} + S_t\,P_t^F b_t^F - P_tT_t,$$

with capital evolving as $K_{t+1} = f\left(\frac{I_t}{K_t}\right) K_t + (1-\delta)K_t$ where $f(\cdot)$ is specified so that $f(0) = 0$, $f(x) = 1$, and $f'(x) = 1$.

Here:

- I_t is investment
- B_t^{US} denotes safe U.S. government bonds,
- B_t^F denotes foreign bonds (with face value in f, converted using S_t/P_t),
- b_t^F denotes the foreign asset position in units of foreign goods: $b_t^F \equiv \frac{B_t^F}{P_t^F}$
- r_t^{US} and r_t^F are the yields on domestic and foreign bonds respectively,
- T_t represents lump-sum taxes.
- ψ_t is the convenience yield on domestic bonds.
- HM_t represents home inputs.
- IM_t represents foreign goods (imports).

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Big[\ln C_t - \xi \ln(1 - H_t) + \phi \ln L_t$$
$$+ \lambda_t \big(W_tH_t + R_{k,t}K_{t-1} + M_t - M_{t-1} + B_t^{US} + S_tP_t^F b_t^F - P_tT_t - P_tC_t - P_tI_t - \tfrac{B_{t+1}^{US}}{R_t^{US}} - \tfrac{S_tP_t^F b_{t+1}^F}{R_t^F} \big) \Big].$$

$$HM_t = (1-\lambda)\left(\frac{P_t^H}{P_t}\right)^{-\varepsilon} C_t, \tag{4}$$

$$IM_t = \lambda\left(\frac{P_t^F}{P_t}\right)^{-\varepsilon} C_t \tag{5}$$

$$
\begin{aligned}
[C_t]: \quad & \frac{1}{C_t} = \lambda_t P_t \\
[H_t]: \quad & \xi \frac{1}{1 - H_t} = \lambda_t W_t \\
[B_t^{US}]: \quad & \lambda_t \frac{1}{R_t^{US}} = \beta E_t[\lambda_{t+1}] \\
[b_t^F]: \quad & \lambda_t \frac{S_t P_t^F}{R_t^F} = \beta E_t[\lambda_{t+1} S_{t+1} P_{t+1}^F]
\end{aligned}
$$

Therefore we have:

$$
\begin{aligned}
[B_t^{US}] \quad & 1 = \beta E_t \left[\frac{\lambda_{t+1}}{\lambda_t} R_t^{US} \right] \\
[b_t^F] \quad & 1 = \beta E_t \left[\frac{\lambda_{t+1} S_{t+1} P_{t+1}^F}{\lambda_t S_t P_t^F} R_t^F \right]
\end{aligned}
$$

Derivation of the Log-Linearized Convenience Yield

1. Setup

Define:

$$b_t \equiv \frac{B_t^{US}}{P_t} \quad \text{(real bond holdings)}$$

Then:

$$L_t = \left[(1 - \theta) M_t^{\frac{\zeta - 1}{\zeta}} + \theta b_t^{\frac{\zeta - 1}{\zeta}} \right]^{\frac{\zeta}{\zeta - 1}}$$

2. First-Order Condition with Respect to Bonds

The marginal utility of bond holdings is:

$$\psi_t \equiv \frac{\partial U}{\partial b_t} = \phi \cdot \frac{1}{L_t} \cdot \frac{\partial L_t}{\partial b_t}$$

To compute $\partial L_t / \partial b_t$, write:

$$L_t = Z_t^{\frac{\zeta}{\zeta - 1}}, \quad \text{where} \quad Z_t = (1 - \theta) M_t^{\frac{\zeta - 1}{\zeta}} + \theta b_t^{\frac{\zeta - 1}{\zeta}}$$

Then:

$$\frac{\partial L_t}{\partial b_t} = \frac{\zeta}{\zeta - 1} Z_t^{\frac{1}{\zeta - 1}} \cdot \theta \cdot \frac{\zeta - 1}{\zeta} b_t^{-\frac{1}{\zeta}} = Z_t^{\frac{1}{\zeta - 1}} \cdot \theta \cdot b_t^{-\frac{1}{\zeta}}$$

Substituting back in:

$$\psi_t = \phi \cdot \frac{1}{L_t} \cdot Z_t^{\frac{1}{\zeta-1}} \cdot \theta \cdot b_t^{-\frac{1}{\zeta}}$$

But since $L_t = Z_t^{\frac{\zeta}{\zeta-1}}$, it follows that:

$$Z_t^{\frac{1}{\zeta-1}} = L_t^{1/\zeta} \quad \Rightarrow \quad \frac{1}{L_t} \cdot Z_t^{\frac{1}{\zeta-1}} = L_t^{-1+1/\zeta}$$

Thus:

$$\psi_t = \phi \cdot \theta \cdot b_t^{-1/\zeta} \cdot L_t^{-1+1/\zeta}$$

3. Log-Linearization

Let steady-state values be denoted with bars:

$$\bar{\psi}, \quad \bar{b}, \quad \bar{L}$$

Define log-deviations:

$$\hat{\psi}_t \equiv \ln \psi_t - \ln \bar{\psi}, \quad \hat{b}_t \equiv \ln b_t - \ln \bar{b}, \quad \hat{L}_t \equiv \ln L_t - \ln \bar{L}$$

Then:

$$\ln \psi_t = \ln(\phi\theta) - \frac{1}{\zeta} \ln b_t + \left(-1 + \frac{1}{\zeta}\right) \ln L_t$$

Taking log deviations:

$$\begin{aligned}
\hat{\psi}_t &= \ln \psi_t - \ln \bar{\psi} \\
&= -\frac{1}{\zeta}(\ln b_t - \ln \bar{b}) + \left(-1 + \frac{1}{\zeta}\right)(\ln L_t - \ln \bar{L}) \\
&= -\frac{1}{\zeta}\hat{b}_t - \left(1 - \frac{1}{\zeta}\right)\hat{L}_t
\end{aligned}$$

4. Log-Linearization of the Liquidity Aggregator

From the CES form, the log-linear approximation is:

$$\hat{L}_t = (1-\theta)\,\hat{M}_t + \theta\,\hat{b}_t$$

Substitute into the expression for $\hat{\psi}_t$:

$$\begin{aligned}
\hat{\psi}_t &= -\frac{1}{\zeta}\hat{b}_t - \left(1 - \frac{1}{\zeta}\right)\left[(1-\theta)\,\hat{M}_t + \theta\,\hat{b}_t\right] \\
&= -(1-\theta)\left(1 - \frac{1}{\zeta}\right)\hat{M}_t - \left[\frac{1}{\zeta} + \theta\left(1 - \frac{1}{\zeta}\right)\right]\hat{b}_t
\end{aligned}$$

Simplifying the coefficients:

$$\hat{\psi}_t = -(1-\theta)\,\hat{M}_t - \left(\theta + \frac{1}{\zeta}\right)\hat{b}_t$$

5. Final Result

$$\boxed{\hat{\psi}_t = -(1-\theta)\,\hat{M}_t - \left(\theta + \frac{1}{\zeta}\right)\hat{b}_t}$$

This is the log-linearized expression for the convenience yield ψ_t, consistent with microfoundations where both money and bonds provide liquidity services.

Capital Accumulation:

For the capital accumulation equation we have

$$\ln K_{t+1} = \ln\left[f\left(\frac{I_t}{K_t}\right)K_t + (1-\delta)K_t\right]$$

Totally differentiating and rearranging terms yields:

$$k_{t+1} = \delta k_t + \left(\frac{I}{K}\right)i_t - \delta k_t + (1-\delta)k_t$$

which gives us:

$$k_{t+1} = \delta i_t + (1-\delta)k_t \tag{6}$$

Tobin's Q:

From the FOC for I_t we have $-\lambda_t + \phi_t f'\left(\frac{I_t}{K_t}\right) = 0$, which entails that $\phi_t = \frac{\lambda_t}{f'(I_t/K_t)}$. Tobin's Q gives us $Q = \frac{\phi_t}{\lambda_t} = \frac{\lambda_t/f'(I_t/K_t)}{\lambda_t}$, which yields:

$$Q_t = \left[f'\left(\frac{I_t}{K_t}\right)\right]^{-1}$$

when we totally differentiate this after taking logs we get:

$$q_t = -\frac{1}{Q}\left[f''\left(\frac{I}{K}\right)\left(\frac{dI_t}{K} - \frac{I}{K^2}\right)dK_t\right]$$

which yields:

$$q_t = -f''(\delta)\delta[i_t - k_t] \tag{7}$$

Aggregate Production Function

Our aggregate production function is specified as:

$$d_t Y_t = A_t K_t^{\alpha} H_t^{\Omega(1-\alpha)} (H_t^e)^{(1-\Omega)(1-\alpha)}$$

where d_t is the standard price dispersion equation. Linearizing it, we get:

$$y_t = a_t + \alpha k_t + \Omega(1-\alpha) h_t \tag{8}$$

Inflation Utilizing the optimal reset price and the price dispersion equations, linearizing them, and then combining conditions gets us our inflation equation:

$$\pi_t = -\frac{(1-\theta)(1-\theta\beta)}{\theta} x_t + \beta E_t \pi_{t+1} \tag{9}$$

where x_t is the real marginal cost.

Net Worth To start, we assume that only the fraction ρ of entrepreneurs survive each period. Those that die "eat" their net worth:

$$C_t^e = (1-\rho) V_t$$

Where V_t represents capital equity in the following manner:

$$V_t = (R_t^K - R_{t-1})(Q_{t-1}K_t - N_{t-1}) + R_{t-1}N_{t-1}$$

While the entrepreneur remains alive, his net worth can be defined as:

$$N_t = \rho V_t + W_t^e$$

Where W_t^e is the entrepreneur's wages. Here, ρ is close to 1 and entreprenurial wages are small. As a result, we can say that:

$$c_t^e = n_t \tag{10}$$

We assume that net worth follows:

$$N_t = \rho[(R_t^k - R_{t-1})Q_{t-1}K_t + \gamma_t(Q_{t-1}K_t - N_t) + R_{t-1}N_{t-1}] + w_t^e$$

with

$$\gamma_t = \tau \int_0^{\bar{\omega}} \omega_t f(\omega_t) R_t^k Q_{t-1} K_t d\omega_t$$

After taking logs and totally differentiating we end up with:

$$n_t = \frac{\rho R K}{R}(r_t^k - r_t) + \rho R(r_{t-1} + n_{t-1}) + \rho \frac{K}{N}\left(\frac{R^k}{R} - 1\right)(r_t^k + q_{t-1} + k_t) + \frac{W^e}{N} w_t^e \tag{11}$$

1.2 Entrepreneurs

The default cutoff is implicitly defined by

$$Z_{t+1}\Big(Q_t K_{t+1} - N_t\Big) = \bar{\omega}_{t+1}\, R^k_{t+1}\, Q_t K_{t+1}\,. \tag{12}$$

Define the leverage ratio as

$$L_t \equiv \frac{Q_t K_{t+1}}{N_t}\,.$$

Then the loan rate satisfies:

$$Z_{t+1} = \bar{\omega}_{t+1}\, R^k_{t+1}\, \frac{L_t}{L_t - 1}\,. \tag{13}$$

The expected entrepreneurial income from obtaining a loan is

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\, \phi(\omega_{t+1})\, d\omega_{t+1}\, R^k_{t+1}\, Q_t K_{t+1} - \Big[1 - \Phi(\bar{\omega}_{t+1})\Big]\, Z_{t+1}\Big(Q_t K_{t+1} - N_t\Big)\,.$$

Using equation 16 to substitute for Z_{t+1}, the expression becomes

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\, \phi(\omega_{t+1})\, d\omega_{t+1}\, R^k_{t+1}\, Q_t K_{t+1} - \Big[1 - \Phi(\bar{\omega}_{t+1})\Big]\, \bar{\omega}_{t+1}\, R^k_{t+1}\, Q_t K_{t+1}\,.$$

Define

$$f(\bar{\omega}_{t+1}) \equiv \int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\, \phi(\omega_{t+1})\, d\omega_{t+1} - \Big[1 - \Phi(\bar{\omega}_{t+1})\Big]\, \bar{\omega}_{t+1}\,.$$

Then the expected entrepreneurial income is

$$f(\bar{\omega}_{t+1})\, R^k_{t+1}\, Q_t K_{t+1}\,.$$

(Definition of $f(\bar{\omega}_{t+1})$ as given above.)
Rewriting in terms of leverage $L_t = \frac{Q_t K_{t+1}}{N_t}$, the firm's expected return becomes

$$f(\bar{\omega}_{t+1})\, R^k_{t+1}\, L_t\,.$$

The lender's expected return from the project is

$$\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\, (1-\mu)\, R^k_{t+1}\, Q_t K_{t+1}\, \phi(\omega_{t+1})\, d\omega_{t+1} + \Big[1 - \Phi(\bar{\omega}_{t+1})\Big]\, Z_{t+1}\Big(Q_t K_{t+1} - N_t\Big)\,.$$

Using equation 16 again to eliminate Z_{t+1}, the lender's expected return becomes

$$\left[(1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\, \phi(\omega_{t+1})\, d\omega_{t+1} + \Big(1 - \Phi(\bar{\omega}_{t+1})\Big)\bar{\omega}_{t+1}\right] \times\; R^k_{t+1}\, Q_t K_{t+1}\,.$$

Define

$$g(\bar{\omega}_{t+1}) \equiv (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\, \phi(\omega_{t+1})\, d\omega_{t+1} + \Big(1 - \Phi(\bar{\omega}_{t+1})\Big)\bar{\omega}_{t+1}\,.$$

Then the lender's return on the loan is

$$R^k_{t+1}\, g(\bar{\omega}_{t+1})\, \frac{L_t}{L_t - 1}\,.$$

The entrepreneur chooses $\bar{\omega}_{t+1}$ and L_t to maximize his expected return:

$$\max_{\bar{\omega}_{t+1},\, L_t} \; E_t\Big[R^k_{t+1}\, f(\bar{\omega}_{t+1})\, L_t\Big]\,,$$

subject to the lender's participation constraint

$$R^k_{t+1}\, g(\bar{\omega}_{t+1})\, \frac{L_t}{L_t - 1} \geq R_t\,.$$

Equivalently, the participation constraint can be written as

$$R^k_{t+1}\, g(\bar{\omega}_{t+1})\, L_t = \big(L_t - 1\big)\, R_t\,.$$

Form the Lagrangian for the entrepreneur's problem:

$$\mathcal{L} = E_t\Bigg\{R^k_{t+1}\, f(\bar{\omega}_{t+1})\, L_t + \Lambda_{t+1}\Big[R^k_{t+1}\, g(\bar{\omega}_{t+1})\, L_t - R_t(L_t - 1)\Big]\Bigg\}\,.$$

Taking the derivative with respect to $\bar{\omega}_{t+1}$ yields:

$$E_t\Big\{R^k_{t+1}\, f'(\bar{\omega}_{t+1})\, L_t + \Lambda_{t+1}\, R^k_{t+1}\, g'(\bar{\omega}_{t+1})\, L_t\Big\} = 0\,.$$

Differentiating with respect to L_t gives:

$$E_t\Big\{R^k_{t+1}\, f(\bar{\omega}_{t+1}) + \Lambda_{t+1}\Big[R^k_{t+1}\, g(\bar{\omega}_{t+1}) - R_t\Big]\Big\} = 0\,.$$

The participation constraint (which is binding) is

$$R^k_{t+1}\, g(\bar{\omega}_{t+1})\, L_t = (L_t - 1)\, R_t\,.$$

At steady state the multiplier satisfies

$$\Lambda = -\frac{f'(\bar{\omega})}{g'(\bar{\omega})}\,.$$

Linearize around the steady state. Define the proportional deviation

$$\hat{\omega}_{t+1} \equiv \frac{d\bar{\omega}_{t+1}}{\bar{\omega}}$$

and let λ_{t+1} denote the deviation of Λ_{t+1}. Then, to first order,

$$\Psi\, \hat{\omega}_{t+1} = \lambda_{t+1}\,,$$

where

$$\Psi = \bar{\omega}\left(\frac{f''(\bar{\omega})}{f'(\bar{\omega})} - \frac{g''(\bar{\omega})}{g'(\bar{\omega})}\right).$$

Linearize to obtain

$$r^k_{t+1} - r_t + l_t + \Theta_f\,\hat{\omega}_{t+1} = \lambda_{t+1}\,,$$

with

$$\Theta_f \equiv \bar{\omega}\,\frac{f'(\bar{\omega})}{f(\bar{\omega})}\,,$$

and where l_t denotes the percentage deviation of leverage.

Linearize to get

$$r^k_{t+1} - r_t + \Theta_g\,\hat{\omega}_{t+1} = \frac{1}{L-1}\,l_t\,,$$

where

$$\Theta_g \equiv \bar{\omega}\,\frac{g'(\bar{\omega})}{g(\bar{\omega})}\,.$$

Now I derive the financial accelerator (FA) leverage–spread equation with a CMR–style risk shock. The derivation begins from the Bernanke–Gertler–Gilchrist (BGG) contracting block (as in Sims' `bbg_ers_notes_final.pdf`) and then introduces a time–varying dispersion parameter (denoted σ_t) in the idiosyncratic return. This extra state leads to an augmentation in the external finance premium equation, resulting in:

$$\boxed{r_{k,t+1} - r_t = -\nu\big[n_t - q_t - k_{t+1}\big] + \chi\,\hat{\sigma}_t,}$$

where $\hat{\sigma}_t$ is the (log) deviation of σ_t from its steady state.

1.2.1 Recap of the BGG Setup (Without Risk Shocks)

In the standard BGG model, we denote (all variables are log–deviations from the steady state):

[label=–]q_t: (log) price of capital, k_{t+1}: capital chosen in period t for use in $t+1$, n_t: entrepreneurs' net worth, $r_{k,t+1}$: ex–post real return on capital, r_t: risk–free real interest rate, $\bar{\omega}_{t+1}$: default cutoff in idiosyncratic return, $L_t \equiv \dfrac{q_t k_{t+1}}{n_t}$: leverage ratio, $f(\bar{\omega}_t)$ and $g(\bar{\omega}_t)$: respectively, the share of returns kept by the entrepreneur and the lender, and λ_{t+1}: the Lagrange multiplier (from the lender's participation constraint).

Sims' notes present the following three linearized conditions (Sims' eqs. (53)–(55)):

$$\Psi\,\hat{\bar{\omega}}_{t+1} = \lambda_{t+1},$$
$$r_{k,t+1} - r_t + \ell_t + \Theta_f\,\hat{\bar{\omega}}_{t+1} = \lambda_{t+1},$$
$$r_{k,t+1} - r_t + \Theta_g\,\hat{\bar{\omega}}_{t+1} = \frac{1}{L-1}\,\ell_t,$$

where we define the leverage wedge as

$$\ell_t \equiv q_t + k_{t+1} - n_t,$$

and $\hat{\bar{\omega}}_{t+1}$ denotes the log–deviation of the cutoff (i.e. $\hat{\bar{\omega}}_{t+1} = d\ln\bar{\omega}_{t+1}$). The parameters Ψ, Θ_f, and Θ_g are steady–state elasticities derived from the functions f and g.

Eliminating $\hat{\bar{\omega}}_{t+1}$ and λ_{t+1} from 53–55 yields the BGG external finance premium:

$$r_{k,t+1} - r_t = -\nu\,\ell_t \quad \text{with} \quad \nu = \frac{\Psi}{\Psi(L-1) - \Theta_f L}.$$

In other words,

$$r_{k,t+1} - r_t = -\nu\Big[n_t - \big(q_t + k_{t+1}\big)\Big].$$

1.3 Introducing the CMR Risk Shock

1.3.1 Time–Varying Dispersion in Idiosyncratic Returns

CMR assume that the idiosyncratic return shock follows a lognormal distribution with a time–varying dispersion parameter σ_t:

$$\omega_{t+1} \sim \exp\big(\sigma_t\,u_{t+1}\big), \qquad u_{t+1} \sim \mathcal{N}(0,1).$$

Its law of motion is given by

$$\log\sigma_t = \rho_\sigma \log\sigma_{t-1} + \varepsilon_t^\sigma.$$

For convenience, define the log deviation (or percentage deviation) of σ_t from its steady state as

$$\hat{\sigma}_t \equiv \log\sigma_t - \log\sigma.$$

1.3.2 Dependence of f and g on σ_t

In the standard derivation, the functions f and g determine the shares of gross returns retained by the entrepreneur and the lender. Now, because the density

$$\phi_t(\omega)$$

depends on σ_t, we have

$$f_t = f(\bar{\omega}_t, \sigma_t), \qquad g_t = g(\bar{\omega}_t, \sigma_t).$$

As a result, the log derivatives pick up extra terms. Define:

$$\Theta_f^\sigma \equiv \frac{\partial \ln f}{\partial \ln\sigma} \quad \text{and} \quad \Theta_g^\sigma \equiv \frac{\partial \ln g}{\partial \ln\sigma}.$$

Similarly, if the multiplier λ_t depends on σ_t, define

$$\Xi \equiv \frac{\partial\lambda}{\partial\ln\sigma}\Big/\frac{\partial\bar{\omega}}{\partial\lambda}.$$

1.4 Derivation of the FA–CMR Equation

1.4.1 Log–Linearized System Including Risk Shock

Re–linearize the three FOCs, retaining first–order terms in $\hat{\sigma}_t$. This yields:

$$\Psi\,\hat{\bar{\omega}}_{t+1} \;+\; \Xi\,\hat{\sigma}_t = \lambda_{t+1},$$

$$r_{k,t+1} - r_t + \ell_t \;+\; \Theta_f\,\hat{\bar{\omega}}_{t+1} \;+\; \Theta_f^{\sigma}\,\hat{\sigma}_t = \lambda_{t+1},$$

$$r_{k,t+1} - r_t \;+\; \Theta_g\,\hat{\bar{\omega}}_{t+1} \;+\; \Theta_g^{\sigma}\,\hat{\sigma}_t = \frac{1}{L-1}\,\ell_t.$$

1.4.2 Eliminating $\hat{\bar{\omega}}_{t+1}$ and λ_{t+1}

1: Solve Equation for λ_{t+1}:

$$\lambda_{t+1} = \Psi\,\hat{\bar{\omega}}_{t+1} + \Xi\,\hat{\sigma}_t.$$

2: Substitute this expression into Equation:

$$r_{k,t+1} - r_t + \ell_t + \Theta_f\,\hat{\bar{\omega}}_{t+1} + \Theta_f^{\sigma}\,\hat{\sigma}_t = \Psi\,\hat{\bar{\omega}}_{t+1} + \Xi\,\hat{\sigma}_t.$$

Rearrange to isolate $\hat{\bar{\omega}}_{t+1}$:

$$(\Theta_f - \Psi)\,\hat{\bar{\omega}}_{t+1} = -\ell_t + (\Xi - \Theta_f^{\sigma})\,\hat{\sigma}_t. \tag{14}$$

3: Substitute λ_{t+1} into Equation:

$$r_{k,t+1} - r_t + \Theta_g\,\hat{\bar{\omega}}_{t+1} + \Theta_g^{\sigma}\,\hat{\sigma}_t = \frac{1}{L-1}\,\ell_t.$$

We now have two equations (and the above) that contain $\hat{\bar{\omega}}_{t+1}$ and $\hat{\sigma}_t$.

4: Solve for $\hat{\bar{\omega}}_{t+1}$:

$$\hat{\bar{\omega}}_{t+1} = \frac{-\ell_t + (\Xi - \Theta_f^{\sigma})\,\hat{\sigma}_t}{\Theta_f - \Psi}.$$

Insert this into the third equation:

$$r_{k,t+1} - r_t + \Theta_g\,\frac{-\ell_t + (\Xi - \Theta_f^{\sigma})\,\hat{\sigma}_t}{\Theta_f - \Psi} + \Theta_g^{\sigma}\,\hat{\sigma}_t = \frac{1}{L-1}\,\ell_t.$$

Multiply both sides by $\Theta_f - \Psi$:

$$(\Theta_f - \Psi)(r_{k,t+1} - r_t) = -\Theta_g\,\ell_t + (\Theta_f - \Psi)\frac{1}{L-1}\,\ell_t + \Big[\Theta_g(\Xi - \Theta_f^{\sigma}) + (\Theta_f - \Psi)\Theta_g^{\sigma}\Big]\,\hat{\sigma}_t.$$

Thus,

$$r_{k,t+1} - r_t = -\frac{\Theta_g - \frac{\Theta_f - \Psi}{L-1}}{\Theta_f - \Psi}\,\ell_t + \frac{\Theta_g(\Xi - \Theta_f^{\sigma}) + (\Theta_f - \Psi)\Theta_g^{\sigma}}{\Theta_f - \Psi}\,\hat{\sigma}_t. \tag{15}$$

1.4.3 Step 5: Use the Steady–State Identity and Define New Parameters

Sims shows that in steady state one obtains an identity relating the elasticities:

$$\Theta_g(L-1) = -\Theta_f.$$

Using this identity to simplify the coefficient on ℓ_t and defining:

$$\nu \equiv \frac{\Psi}{\Psi(L-1) - \Theta_f L}, \quad \chi \equiv \frac{\Xi(L-1) - \Theta_f^\sigma L + \Theta_g^\sigma (L-1)}{\Psi(L-1) - \Theta_f L},$$

and recalling that $\ell_t = q_t + k_{t+1} - n_t$, Equation becomes

$$\boxed{r_{k,t+1} - r_t = -\nu \Big[n_t - q_t - k_{t+1} \Big] + \chi \, \hat{\sigma}_t.}$$

This is the FA–CMR leverage–spread equation with a risk shock. The coefficient ν is identical to its definition in the BGG model; it captures how the leverage ratio affects the external finance premium. The new coefficient χ reflects the sensitivity of the spread to the risk shock $\hat{\sigma}_t$. A positive $\hat{\sigma}_t$ (i.e., an increase in the dispersion of the idiosyncratic shocks) increases the external finance premium, even if leverage is unchanged.

1.5 Foreign Block

Letting S_t denote the nominal exchange rate of the home currency (the price of one unit of foreign currency in terms of the home currency), we have the law of one price:

$$P_t^F = S_t P_t^{F*}$$

Where P_t^F is the domestic-currency price of the foreign good, and P_t^{F*} is the foreign currency price of those goods. We define the prices of the domestic and the foreign goods in terms of the consumer price index P_t:

$$p_t^H = \frac{P_t^H}{P_t}$$

$$p_t^F = \frac{P_t^F}{P_t}$$

Where we indentify p_t^F as the real exchange rate. We can define the terms of trade as:

$$TOT_t = \frac{P_t^H}{P_t^F} \tag{16}$$

Which yields allows us to specify the home country's export demand as:

$$EX_t = TOT_t^{-\sigma^*} WT_t \tag{17}$$

Define the CPI inflation rate, foreign inflation rate, and the rate of change of the nominal exchange rate, respectively, as:

$$\pi_t = \frac{P_t}{P_{t-1}}$$

$$\pi_t^F = \frac{P_t^F}{P_{t-1}^F}$$
$$\sigma_{t+1} = \frac{S_{t+1}}{S_t}$$

Then we have:

$$\pi_t^F = \sigma_t + \pi_t^*$$

Setting foreign-currency import price to be constant ($\pi_t^* = 0$), we get $\pi_t^F = \sigma_t$. In logs, we have $p_t^F = (\ln P_t^F - \ln P_t)$. Therefore,

$$p_t^F - p_{t-1}^F = \pi_t^F - \pi_t$$

This give us the condition:

$$p_t^F - p_{t-1}^F = \sigma_t - \pi_t \tag{18}$$

Using the two Euler equations from the household's problem, we can derive the Uncovered Interest Rate Parity (UIP), starting wth the FOC on B_t^{US}:

$$\frac{1}{C_t P_t} = \beta E_t \left[\frac{1}{C_{t+1} P_{t+1}} R_t^{US} \right]$$

Defining $\frac{P_{t+1}}{P_t} = \Pi_{t+1}$, and using the fact that $R_t^{US} = 1 + r_t^{US} + \psi_t$, by log-linearizing the condition we get:

$$c_t = -r_t^{US} - \psi_t + c_{t+1} + \pi_{t+1}$$

Performing the same procedure in the case of b_t^F yields:

$$c_t - s_t = -r_t^F + c_{t+1} - s_{t+1} - \pi_{t+1}^F + \pi_{t+1}$$

Combining these two conditions gives us UIP:

$$s_{t+1} - s_t = r_t^{US} - r_t^F + \psi_t - \pi_{t+1}^F \tag{19}$$

1.6 Wage Inflation

Under Calvo wage setting, a household that reoptimizes in period t chooses W_t^* to maximize

$$\sum_{j=0}^{\infty} (\beta\theta_w)^j E_t \Big[\Lambda_{t,t+j} \big(U_{C,t+j}\, W_t^*\, H_{t+j|t} - U_{H,t+j}\, H_{t+j|t} \big) \Big],$$

subject to

$$H_{t+j|t} = \left(\frac{W_t^*}{W_{t+j}} \right)^{-\varphi_w} H_{t+j}.$$

Upon taking FOCs, we get the standar optimal reset-wage condition:

$$W_t^* = \frac{\varphi_w}{\varphi_w - 1} \frac{\sum_{j=0}^{\infty} (\beta\theta_w)^j E_t[\Lambda_{t,t+j} U_{C,t+j} H_{t+j} W_{t+j}^{\varphi_w} mrs_{t+j}]}{\sum_{j=0}^{\infty} (\beta\theta_w)^j E_t[\Lambda_{t,t+j} U_{C,t+j} H_{t+j} W_{t+j}^{\varphi_w - 1}]}$$

Linearizing the first-order condition around a symmetric steady state and defining

$$w_t \equiv \ln W_t - \ln P_t, \quad wgap_t \equiv w_t - MRS_t,$$

Where $MRS_t = \frac{U_{H,t}}{U_{C,t}}$.

This yields

$$w_t^* - w_t = -\underbrace{\frac{1-\beta\,\theta_w}{\theta_w}}_{\zeta_w}\sum_{j=0}^{\infty}(\beta\,\theta_w)^j E_t\big[wgap_{t+j}\big]. \tag{20}$$

1.6.1 Aggregation to Wage Inflation

The Calvo law of motion for aggregate wages implies

$$w_t = \theta_w\, w_{t-1} + (1-\theta_w)\, w_t^*,$$

so gross wage inflation is

$$\pi_t^W \equiv w_t - w_{t-1} = (1-\theta_w)\big(w_t^* - w_t\big). \tag{21}$$

Substitute (19) into (20):

$$\pi_t^W = -\,(1-\theta_w)\,\zeta_w \sum_{j=0}^{\infty}(\beta\,\theta_w)^j E_t\big[wgap_{t+j}\big].$$

1.6.2 Forming the NKPC

Compute the one-period-ahead expectation:

$$\beta\, E_t[\pi_{t+1}^W] = -\,(1-\theta_w)\,\zeta_w \sum_{j=1}^{\infty}(\beta\,\theta_w)^j E_t\big[wgap_{t+j}\big].$$

Subtract from π_t^W:

$$\pi_t^W - \beta\, E_t[\pi_{t+1}^W] = (1-\theta_w)\,\zeta_w\, wgap_t.$$

1.6.3 Final Form

Define

$$\kappa_w \equiv (1-\theta_w)\,\zeta_w = \frac{(1-\theta_w)\,(1-\beta\,\theta_w)}{\theta_w}.$$

Thus the final wage inflation rate:

$$\boxed{\pi_t^W \;=\; \beta\, E_t[\pi_{t+1}^W] \;-\; \kappa_w\, wgap_t.}$$

1.7 Log-Linearized Equilibrium Conditions

$$
\begin{aligned}
y_t &= hm_t + \gamma_y g_t + ex_t + i_t && (1)\\
c_t &= -r_t + E_t[c_{t+1}] - \psi_t + \pi_{t+1} && (2)\\
ce_t &= n_t && (3)\\
E_t[rk_{t+1}] - r_t &= -\nu(n_t - q_t - k_t) + \chi\, \sigma_t^s && (4)\\
rk_t &= (1-\varepsilon)(y_t - k_{t-1} - x_t) + \varepsilon\, q_t - q_{t-1} && (5)\\
q_t &= \varphi(i_t - k_{t-1}) && (6)\\
y_t &= a_t + \alpha\, k_{t-1} + (1-\alpha)\,\Omega\, h_t && (7)\\
y_t - h_t - x_t - c_t &= \eta^{-1} h_t && (8)\\
k_t &= \delta\, i_t + (1-\delta)\, k_{t-1} && (9)\\
n_t &= \gamma\, RR_{ks}(rk_t - r_{t-1}) + r_{t-1} + n_{t-1} + e_t^n && (10)\\
rn_t &= \theta_i\, rn_{t-1} + (1-\theta_i)\left(\theta_\pi\, \pi_t^{(4)} + \theta_y\, y_t^{\text{gap}}\right) + s_{rn}\, e_t^{rn} && (11)\\
rn_t &= r_t + E_t[\pi_{t+1}] && (12)\\
a_t &= \rho_a\, a_{t-1} + s_a\, e_t^a && (13)\\
g_t &= \rho_g\, g_{t-1} + s_g\, e_t^g && (14)\\
\ell_t &= q_t + k_t - n_t && (15)\\
hm_t &= c_t - \sigma_{cf}\, p_t^H && (16)\\
p_t^H - p_{t-1}^H &= \pi_t^H - \pi_t && (17)\\
\pi_t &= (1-\tau_{\text{open}})\, \pi_t^H + \tau_{\text{open}}\, \sigma_t && (18)\\
\pi_t^H &= \beta\, E_t[\pi_{t+1}^H] - \kappa_p\, x_t && (19)\\
\sigma_t &= s_t - s_{t-1} && (20)\\
s_t - s_{t-1} &= r_{t-1} - r_{t-1}^F + \psi_{t-1} - \pi_t^F && (21)\\
ex_t &= -\varepsilon\, \text{tot}_t + wt_t && (22)\\
\text{tot}_t &= p_t^H - p_t^F && (23)\\
p_t^F - p_{t-1}^F &= \sigma_t - \pi_t && (24)\\
wt_t &= \rho_{wt}\, wt_{t-1} + \psi_{wt}\, \sigma_t + \sigma_{wt}\, e_t^{wt} && (25)\\
\sigma_t^s &= \rho_\sigma\, \sigma_{t-1}^s + s_\sigma\, e_t^\sigma && (26)\\
im_t &= c_t - \sigma_{cf}\, p_t^F && (27)\\
\pi_t^W &= \beta\, E_t[\pi_{t+1}^W] - \kappa_w\, wgap_t && (28)\\
wgap_t - wgap_{t-1} &= \pi_t^W - \pi_t && (29)\\
\psi_t &= -(1-\theta_\ell)\, M_t - \left(\theta_\ell + \frac{1}{\zeta_p}\right) b_t^{US} && (30)\\
L_t &= (1-\theta_\ell)\, M_t + \theta_\ell\, b_t^{US} && (31)\\
M_t &= \rho_M\, M_{t-1} + e_t^M && (32)
\end{aligned}
$$

$$b_t^{US} = \rho_b \, b_{t-1}^{US} + (1-\rho_b)\,(\gamma_y \, g_t + (1-\gamma_y)\, y_t) \tag{33}$$

2 Notes

Expenditure Minimization and Relative Demand Derivation

We assume households consume a CES composite good C_t made up of domestically produced goods HM_t and imported goods IM_t, according to the Armington aggregator:

$$C_t = \left[(1-\lambda)^{1/\varepsilon}(HM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \qquad 0<\lambda<1,\ \varepsilon>1 \tag{1}$$

The household chooses $\{HM_t, IM_t\}$ to minimize the total cost of achieving a given C_t, taking prices as given.

Expenditure Minimization Problem

Let $P_t^H = 1$ (normalized domestic price) and let P_t^F denote the relative price of imported goods (in domestic good units). The problem is:

$$\begin{aligned} \min_{HM_t, IM_t} \quad & HM_t + P_t^F \cdot IM_t \\ \text{subject to:} \quad & C_t = \left[(1-\lambda)^{1/\varepsilon}(HM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \end{aligned}$$

Lagrangian

Form the Lagrangian function with multiplier μ_t:

$$\begin{aligned} \mathcal{L} = {} & HM_t + P_t^F IM_t \\ & - \mu_t \left(\left[(1-\lambda)^{1/\varepsilon}(HM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} - C_t \right) \end{aligned}$$

First-Order Conditions

Take the partial derivative with respect to HM_t:

$$\frac{\partial \mathcal{L}}{\partial HM_t} = 1 - \mu_t \cdot \frac{\varepsilon}{\varepsilon-1}\left[(1-\lambda)^{1/\varepsilon}(HM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}} \cdot (1-\lambda)^{1/\varepsilon} \cdot (HM_t)^{-1/\varepsilon} = 0$$

Similarly, for IM_t:

$$\frac{\partial \mathcal{L}}{\partial IM_t} = P_t^F - \mu_t \cdot \frac{\varepsilon}{\varepsilon-1}\left[(1-\lambda)^{1/\varepsilon}(HM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}} \cdot \lambda^{1/\varepsilon} \cdot (IM_t)^{-1/\varepsilon} = 0$$

Relative Demand Equation

Divide the FOCs to eliminate μ_t and the bracketed composite term:

$$\frac{1}{P_t^F} = \frac{(1-\lambda)^{1/\varepsilon}}{\lambda^{1/\varepsilon}} \cdot \left(\frac{IM_t}{HM_t}\right)^{1/\varepsilon} \quad \Rightarrow \quad \frac{HM_t}{IM_t} = \frac{1-\lambda}{\lambda} \cdot \left(\frac{1}{P_t^F}\right)^{\varepsilon}$$

$$\boxed{\frac{HM_t}{IM_t} = \frac{1-\lambda}{\lambda}\left(\frac{1}{P_t^F}\right)^{\varepsilon}} \tag{2}$$

This equation shows that the ratio of domestic to imported goods depends on their relative price and the elasticity of substitution ε.

We begin with the CES Armington aggregator for the consumption bundle:

$$C_t = \left[(1-\lambda)^{1/\varepsilon}(HM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{1}$$

We also have the relative demand condition (derived from FOCs):

$$\frac{HM_t}{IM_t} = \frac{1-\lambda}{\lambda}\left(\frac{1}{P_t^F}\right)^{\varepsilon} \quad \Rightarrow \quad HM_t = \frac{1-\lambda}{\lambda}\left(\frac{1}{P_t^F}\right)^{\varepsilon} IM_t \tag{2}$$

Step: Plug (2) into (1) to eliminate HM_t

Substitute equation (2) into equation (1):

$$\begin{aligned} C_t &= \left[(1-\lambda)^{1/\varepsilon}\left(\frac{1-\lambda}{\lambda}\cdot\frac{1}{P_t^F}\right)^{\frac{\varepsilon-1}{\varepsilon}}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \left[(IM_t)^{\frac{\varepsilon-1}{\varepsilon}}\cdot\left\{(1-\lambda)^{1/\varepsilon}\left(\frac{1-\lambda}{\lambda}\cdot\frac{1}{P_t^F}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}\right\}\right]^{\frac{\varepsilon}{\varepsilon-1}} \\ &= IM_t\cdot\left[(1-\lambda)^{1/\varepsilon}\left(\frac{1-\lambda}{\lambda}\cdot\frac{1}{P_t^F}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}\right]^{\frac{\varepsilon}{\varepsilon-1}} \end{aligned}$$

Solve for IM_t

$$IM_t = \frac{C_t}{\left[(1-\lambda)^{1/\varepsilon}\left(\frac{1-\lambda}{\lambda}\cdot\frac{1}{P_t^F}\right)^{\frac{\varepsilon-1}{\varepsilon}} + \lambda^{1/\varepsilon}\right]^{\frac{\varepsilon}{\varepsilon-1}}}$$

Define the price index P_t

From expenditure duality, the ideal price index for the aggregator is:

$$P_t = \left[(1-\lambda)(P_t^H)^{1-\varepsilon} + \lambda(P_t^F)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \tag{3}$$

If $P_t^H = 1$, this simplifies to:

$$P_t = \left[(1-\lambda) + \lambda(P_t^F)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

Final Result: Level Demands

We now obtain the Marshallian demand equations:

$$\boxed{HM_t = (1-\lambda)\left(\frac{P_t^H}{P_t}\right)^{-\varepsilon} C_t, \qquad IM_t = \lambda\left(\frac{P_t^F}{P_t}\right)^{-\varepsilon} C_t} \tag{4}$$

Below is a simulation of a one standard deviation risk shock.

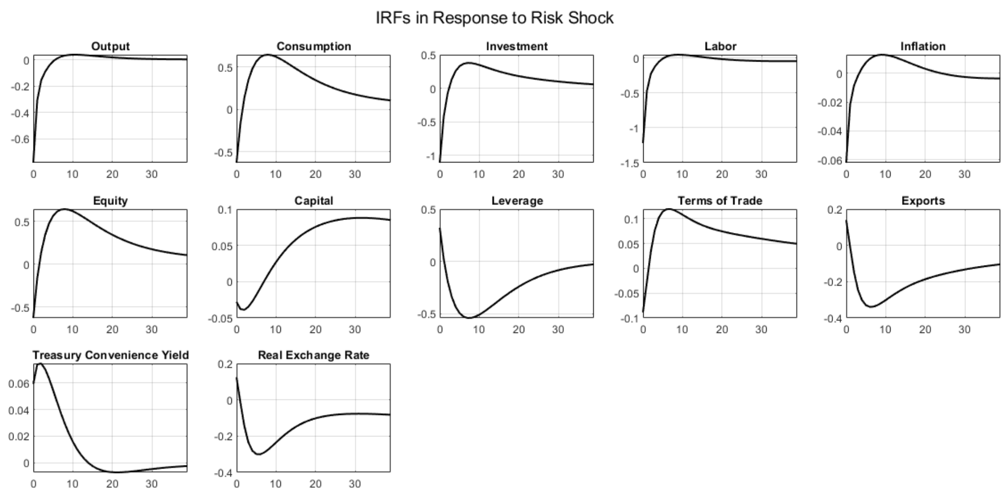

Figure 1: Model IRFs

IRFs of model variables to a one standard deviation risk shock.